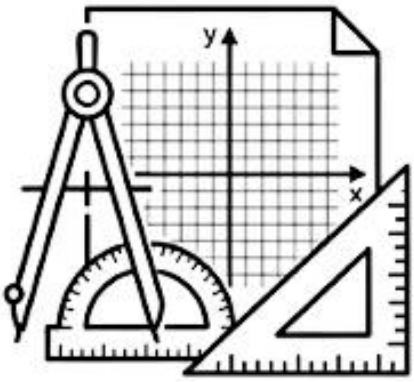
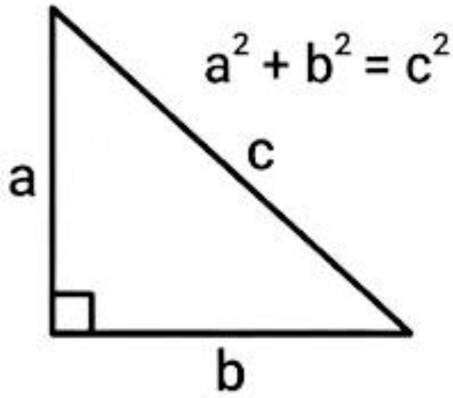
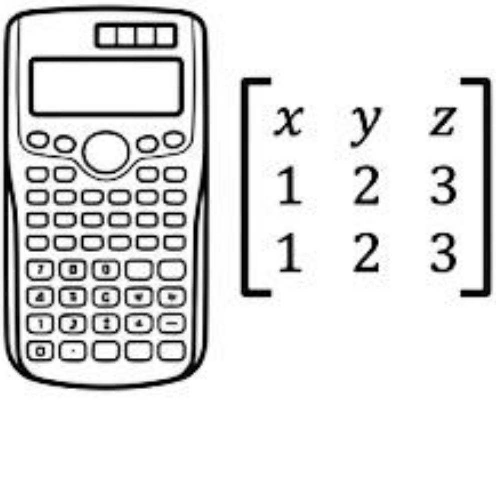
FREE NOTES

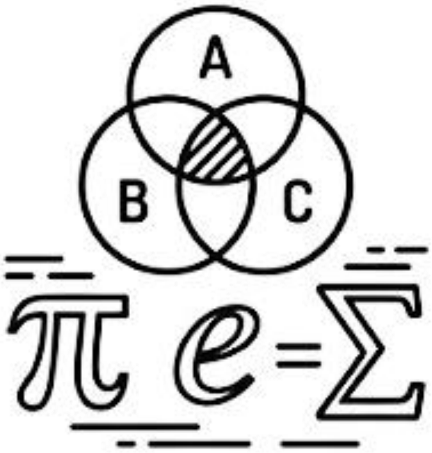
MATHEMATICS (311)

CHAPTERWISE NOTES









MATHEMATICS

Sl. No.	Module	Chapters (Public Examination)	Marks
1	Module 4: Co-ordinate Geometry	L-13 Cartesian System of Rectangular Co-ordinates; L-14 Straight Lines; L-15 Circles; L-16 Conic Sections	15
2	Module 6: Algebra-II	L-20 Matrices; L-21 Determinants; L-22 Inverse of a Matrix and its Applications	17
3	Module 9: Vectors and Three-Dimensional Geometry	L-33 Introduction to Three-Dimensional Geometry; L-34 Vectors; L-35 Plane; L-36 Straight Line	17
4	Module 10: Linear Programming and Mathematical Reasoning	L-37 Linear Programming; L-38 Mathematical Reasoning	9

Component	Details	Marks
Public Exam (Selected Modules 4,6,9,10)	Total Chapters: 13	58
Practical Exam	NA	0
TMA	Tutor Marked Assignment	20
Final Possible Marks		**78 Marks**

Table of Contents

1

Cartesian System of Rectangular Co-ordinates

Introduction

Coordinate geometry is a crucial branch of mathematics where geometry is studied using algebra. In this chapter, we will explore the **rectangular coordinate system**, distance formula, section formula, and the slope of lines. This establishes a strong mathematical foundation for advanced concepts.

Rectangular Coordinate Axes

- Two mutually perpendicular lines (x-axis and y-axis) intersecting at the **Origin** (0,0) form the rectangular coordinate system.
- These axes divide the plane into four distinct regions known as **Quadrants**.

Distance between two points

- The distance between any two points $P(x_1, y_1) and\ Q(x_2, y_2)$ in a plane is calculated using the **Distance Formula**:

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Section Formula

When a point $R(x, y)$ divides the line segment joining two points $P(x_1, y_1) and\ Q(x_2, y_2)$ in the ratio $m_1 : m_2$.

- **Internal Division:**

$$x = \frac{m_1x_2 + m_2x_1}{m_1 + m_2} \quad , \quad y = \frac{m_1y_2 + m_2y_1}{m_1 + m_2}$$

- **External Division:**

$$x = \frac{m_1x_2 - m_2x_1}{m_1 - m_2} \quad , \quad y = \frac{m_1y_2 - m_2y_1}{m_1 - m_2}$$

- **Mid-point Formula:**

$$x = \frac{x_1 + x_2}{2} \quad , \quad y = \frac{y_1 + y_2}{2}$$

Area of a Triangle

- If the vertices of a triangle are $A(x_1, y_1), B(x_2, y_2), and\ C(x_3, y_3)$, its area is found using a specific formula.
- **Formula:**

$$Area = \frac{1}{2}|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)|$$

Condition of Collinearity of Three Points

- Three or more points are called **Collinear** if they lie exactly on the same straight line.
- For three points to be collinear, the area of the triangle formed by them must be exactly zero (0).
- **Formula Condition:**

$$x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) = 0$$

Inclination and Slope of a Line

- **Inclination (θ):** The angle made by a line with the positive direction of the x-axis measured in the anticlockwise direction.
- **Slope (m):** The tangent of the inclination angle of a line is called its slope.
- **Formula:**

$$m = \tan\theta$$

Slope of a line joining two distinct points

- The slope of a line passing through two given points $(x_1, y_1) and\ (x_2, y_2)$ can be calculated without knowing the angle of inclination.
- **Formula:**

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Conditions for Parallelism and Perpendicularity of Lines

- Let the slopes of two lines be $m_1\ and\ m_2$.
- **Parallel Lines:** Two lines are parallel if their slopes are exactly equal.

$$m_1 = m_2$$

- **Perpendicular Lines:** Two lines are perpendicular if the product of their slopes is -1.

$$m_1 \times m_2 = -1$$

Intercepts made by a Line on Axes

- The distances cut off by a straight line on the x-axis and y-axis from the origin are called **Intercepts**.
- If the x-intercept is a and the y-intercept is b, the line intersects the axes at points (a, 0) and (0, b).

Angle between Two Lines

- The acute angle θ between two intersecting lines with slopes m_1 *and* m_2 is determined using their respective slopes.
- **Formula:**

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

Shifting of Origin

- The process of moving the origin $(0,0)$ to a new point (h,k)without changing the orientation or direction of the coordinate axes is called **Shifting of Origin**.
- Let (x,y) be the old coordinates and (X,Y) be the new coordinates.
- **Formula:**

$$x = X + h \quad \text{and} \quad y = Y + k$$

TOP 5 QUESTIONS

Q1. Find the distance between the points $P(2,-3)$ *and* $Q(5,1)$.

Answer- Using the Distance Formula:

$$PQ = \sqrt{(5-2)^2 + (1-(-3))^2}$$

$$PQ = \sqrt{(3)^2 + (4)^2}$$

$$PQ = \sqrt{9+16} = \sqrt{25}$$

PQ = 5 units.

Q2. Find the coordinates of the point which divides the line segment joining the points $(4, -3)$ and $(8, 5)$internally in the ratio $3:1$.

Answer-

Using Section Formula for x:

$$x = \frac{(3 \times 8) + (1 \times 4)}{3 + 1} = \frac{24 + 4}{4} = 7$$

Using Section Formula for y:

$$y = \frac{(3 \times 5) + (1 \times -3)}{3 + 1} = \frac{15 - 3}{4} = 3$$

The coordinates of the point are $(7, 3)$.

Q3. Check whether the points $(1, 5), (2, 3) and (-2, -11)$ are collinear.

Answer-

Using Area of Triangle formula:

$$\frac{1}{2}|1(3 - (-11)) + 2(-11 - 5) + (-2)(5 - 3)|$$

$$= \frac{1}{2}|1(14) + 2(-16) - 2(2)|$$

$$= \frac{1}{2}|14 - 32 - 4| = \frac{1}{2}| - 22| = 11$$

Since the area is 11 (not 0), the points are not collinear.

Q4. Find the slope of the line passing through the points $(3, -2) and (1, 4)$.

Answer-

Using the slope formula:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{4 - (-2)}{1 - 3}$$

$$m = \frac{4 + 2}{-2} = \frac{6}{-2} = -3$$

The slope of the line is -3.

Q5. Find the acute angle between two lines whose slopes are $\frac{1}{2}$ *and* 3.

Answer-

Using the angle formula:

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$\tan\theta = \left|\frac{\frac{1}{2} - 3}{1 + (\frac{1}{2})(3)}\right| = \left|\frac{-\frac{5}{2}}{\frac{5}{2}}\right|$$

$$\tan\theta = |-1| = 1$$

Since $\tan\theta = 1$, the angle $\theta = 45^\circ$.

2

Straight Lines

Introduction

In the previous chapter, we explored the basics of coordinate geometry. In this chapter, we will study the algebraic representation of **straight lines**, exploring their equations in various standard forms, the distance of a point from a given line, and the conditions for families of **parallel and perpendicular lines**.

Straight line parallel to an axis

- A straight line parallel to the x-axis is at a constant distance from it.
- A straight line parallel to the y-axis is at a constant distance from it.
- **Formula (Parallel to x-axis):**

$$y = a$$

(where a is the constant distance from the x-axis)

- **Formula (Parallel to y-axis):**

$$x = b$$

(where b is the constant distance from the y-axis)

Straight line in various standard forms

- A straight line can be represented in different algebraic forms depending on the given parameters like slope, intercepts, or passing points.
- **Slope-Intercept Form:** Equation of a line with slope m and y-intercept c.

$$y = mx + c$$

- **Point-Slope Form:** Equation of a line passing through a point (x_1, y_1) with slope m.

$$y - y_1 = m(x - x_1)$$

- **Two-Point Form:** Equation of a line passing through two points $(x_1, y_1) and\ (x_2, y_2)$.

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

- **Intercept Form:** Equation of a line making intercepts a and b on the x and y axes respectively.

$$\frac{x}{a}+\frac{y}{b}=1$$

- **Normal Form (Perpendicular Form):** Equation of a line where p is the length of the perpendicular from the origin and α is the angle it makes with the x-axis.

$$x\cos\alpha+y\sin\alpha=p$$

General equation of first degree in two variables

- Any general equation of the first degree in x and y always represents a straight line.
- **General Equation:**

$$Ax+By+C=0$$

- **Slope of the General Line:**

$$m=-\frac{A}{B}$$

- **x-intercept and y-intercept:**

$$x\text{-intercept}=-\frac{C}{A},\quad y\text{-intercept}=-\frac{C}{B}$$

Distance of a given point from a given line

- The perpendicular distance d of a point $P(x_1,y_1)$from a line $Ax + By + C = 0$ is the shortest distance between them.
- **Formula:**

$$d=\frac{|Ax_1+By_1+C|}{\sqrt{A^2+B^2}}$$

- **Distance between two parallel lines** $Ax + By + C_1 = 0$ *and* $Ax + By + C_2 = 0$:

$$d=\frac{|C_1-C_2|}{\sqrt{A^2+B^2}}$$

Equation of Parallel or Perpendicular lines

Parallel Line: A line parallel to $Ax + By + C = 0$ has the same slope but a different constant.

- **Equation:**

$$Ax + By + k = 0$$

(where k is a constant)

Perpendicular Line: A line perpendicular to $Ax + By + C = 0$ swaps the coefficients of x and y, changing one sign.

- **Equation:**

$$Bx - Ay + k = 0$$

(where k is a constant)

Equation of family of lines passing through the point of intersection of two lines

- The infinite number of lines passing through the intersection point of two given lines forms a **family of lines**.
- If $L_1 : A_1x + B_1y + C_1 = 0$ *and* $L_2 : A_2x + B_2y + C_2 = 0$ are two intersecting lines, the equation of the family of lines passing through their intersection is given by adding them with a parameter k.
- **Formula:**

$$(A_1x + B_1y + C_1) + k(A_2x + B_2y + C_2) = 0$$

TOP 5 QUESTIONS

Q1. Find the equation of a line passing through the point $(2, -3)$ and having a slope of 4.

Answer-

Using the Point-Slope Form:

$$y - y_1 = m(x - x_1)$$

$$y - (-3) = 4(x - 2)$$

$$y + 3 = 4x - 8$$

4x - y - 11 = 0

Q2. Find the intercepts made by the line $3x + 4y = 12$ on the coordinate axes.

Answer-

Dividing the entire equation by 12 to convert it to Intercept Form:

$$\frac{3x}{12} + \frac{4y}{12} = \frac{12}{12}$$

$$\frac{x}{4} + \frac{y}{3} = 1$$

The x-intercept is 4 and the y-intercept is 3.

Q3. Find the perpendicular distance of the point $(3, -2)$ from the line $4x - 3y - 8 = 0$.

Answer-

Using the Distance Formula:

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$$

$$d = \frac{|4(3) - 3(-2) - 8|}{\sqrt{4^2 + (-3)^2}}$$

$$d = \frac{|12 + 6 - 8|}{\sqrt{16 + 9}} = \frac{|10|}{\sqrt{25}}$$

d = 2 units.

Q4. Find the equation of a line parallel to $2x - 5y + 7 = 0$ and passing through the point $(1, 4)$.

Answer-

The equation of a line parallel to $2x - 5y + 7 = 0$ is $2x - 5y + k = 0$.

Since it passes through $(1, 4)$, substitute $x = 1, y = 4$:

$$2(1) - 5(4) + k = 0$$

$$2 - 20 + k = 0$$

$$k = 18$$

The equation is $2x - 5y + 18 = 0$.

Q5. Find the distance between the parallel lines $3x + 4y - 5 = 0$ and $3x + 4y + 10 = 0$.

Answer-

Using the parallel lines distance formula:

$$d = \frac{|C_1 - C_2|}{\sqrt{A^2 + B^2}}$$

$$d = \frac{|-5 - 10|}{\sqrt{3^2 + 4^2}}$$

$$d = \frac{|-15|}{\sqrt{9 + 16}} = \frac{15}{5}$$

d = 3 units.

3

Circles

Introduction

A **circle** is one of the most fundamental shapes in coordinate geometry. In this chapter, we will study the algebraic equation of a circle in various forms, including the standard form, general form, and diametric form, and learn how to calculate its center and radius from these equations.

Equation of a Circle in Standard Form

- A **Circle** is the locus of a point which moves in a plane such that its distance from a fixed point remains constant.
- The fixed point is called the **Center** and the constant distance is called the **Radius**.
- If the center of the circle is (h, k) and its radius is r, the equation is written in **Standard Form** (or Central Form).
- **Formula:**

$$(x - h)^2 + (y - k)^2 = r^2$$

- If the center is located exactly at the **Origin** $(0, 0)$, the equation simplifies.
- **Formula:**

$$x^2 + y^2 = r^2$$

General Equation of a Circle

- Upon expanding the standard form, we get the algebraic representation known as the **General Equation of a Circle**.
- It is a second-degree equation in x and y where the coefficients of x^2 and y^2 are exactly equal, and there is no xy term.
- **General Equation:**

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

- For this general equation, the coordinates of the **Center** are determined by taking half the coefficients of x and y with opposite signs.
- **Center Formula:**

$$(-g, -f)$$

- The **Radius** r of the circle is calculated using the constants $g, f, and\ c$.
- **Radius Formula:**

$$r = \sqrt{g^2 + f^2 - c}$$

Equation of a Circle in Diametric Form

- When the coordinates of the two end points of a diameter of the circle are given, we can directly find its equation without finding the center or radius first.
- Let the given end points of the diameter be (x_1, y_1) and (x_2, y_2).
- This formula is based on the geometric property that the angle subtended by a diameter at any point on the semicircle is a right angle.
- **Formula:**

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$

TOP 5 QUESTIONS

Q1. Find the equation of the circle whose center is $(2, -3)$ and radius is 5.

Answer-

Using the Standard Form:

$$(x-h)^2 + (y-k)^2 = r^2$$

$$(x-2)^2 + (y-(-3))^2 = 5^2$$

$$(x-2)^2 + (y+3)^2 = 25$$

$$x^2 - 4x + 4 + y^2 + 6y + 9 = 25$$

$x^2 + y^2 - 4x + 6y - 12 = 0$

Q2. Find the center and radius of the circle $x^2 + y^2 - 6x + 4y - 12 = 0$.

Answer-

Comparing with the General Equation $x^2 + y^2 + 2gx + 2fy + c = 0$:

$$2g = -6 \implies g = -3$$

$$2f = 4 \implies f = 2$$

$$c = -12$$

Center $(-g,-f)$ is $(3,-2)$.

Radius $r = \sqrt{g^2+f^2-c} = \sqrt{(-3)^2+2^2-(-12)}$

$$r = \sqrt{9+4+12} = \sqrt{25}$$

Center = (3, -2) and Radius = 5 units.

Q3. Find the equation of the circle passing through the origin $(0,0)$ and having its center at $(3,4)$.

Answer-

Since the circle passes through $(0,0)$, the radius is the distance from the center $(3,4)$ to the origin.

$$r = \sqrt{(3-0)^2+(4-0)^2} = \sqrt{9+16} = \sqrt{25} = 5$$

Using Standard Form:

$$(x-3)^2+(y-4)^2 = 5^2$$

$$x^2-6x+9+y^2-8y+16 = 25$$

$x^2 + y^2 - 6x - 8y = 0$

Q4. Find the equation of the circle where the end points of the diameter are $(1,2)$ and $(3,4)$.

Answer-

Using the Diametric Form:

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2) = 0$$

$$(x-1)(x-3)+(y-2)(y-4) = 0$$

$$(x^2-3x-x+3)+(y^2-4y-2y+8) = 0$$

$$x^2-4x+3+y^2-6y+8 = 0$$

$x^2 + y^2 - 4x - 6y + 11 = 0$

Q5. Find the radius of the circle $2x^2 + 2y^2 - 8x + 10y - 3 = 0$.

Answer-

Divide the entire equation by 2 to make the coefficients of x^2 and y^2 equal to 1:

$$x^2 + y^2 - 4x + 5y - \frac{3}{2} = 0$$

Here, $g = -2$, $f = \frac{5}{2}$, $c = -\frac{3}{2}$.

$$r = \sqrt{(-2)^2 + \left(\frac{5}{2}\right)^2 - \left(-\frac{3}{2}\right)}$$

$$r = \sqrt{4 + \frac{25}{4} + \frac{6}{4}} = \sqrt{\frac{16 + 25 + 6}{4}} = \sqrt{\frac{47}{4}}$$

Radius $= \frac{\sqrt{47}}{2}$ **units.**

4

Conic Sections

Introduction

Conic sections are fundamental geometric curves obtained by intersecting a right circular cone with a plane. In this chapter, we will study the definitions, standard equations, and key properties of the parabola, ellipse, and hyperbola. These mathematical curves have extensive applications in physics, astronomy, and engineering.

Sections of a Cone

- When a plane intersects a double right circular cone, different 2-dimensional curves are formed called **Conic Sections**.
- Depending on the specific angle of intersection between the plane and the cone's axis, the resulting sections are a **Circle**, **Ellipse**, **Parabola**, or **Hyperbola**.

Parabola

- A **Parabola** is the locus of a point in a plane whose distance from a fixed point (focus) is exactly equal to its distance from a fixed straight line (directrix).
- **Standard Equation (Rightward Parabola):**

$$y^2 = 4ax$$

- **Coordinates of the Focus:**

$$(a, 0)$$

- **Equation of the Directrix:**

$$x = -a$$

- The **Latus Rectum** is the chord passing through the focus and perpendicular to the axis of the parabola.
- **Length of Latus Rectum:**

$$Latus\ Rectum = 4a$$

Ellipse

- An **Ellipse** is the locus of a point whose sum of distances from two fixed points (foci) is always constant.
- **Standard Equation:**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (a > b)$$

- **Eccentricity (e):** The ratio of the distance from the center to a focus and the distance from the center to a vertex. For an ellipse, $e < 1$.
- **Formula for Eccentricity:**

$$e = \sqrt{1 - \frac{b^2}{a^2}}$$

- **Coordinates of the Foci:**

$$(\pm ae, 0)$$

- **Length of Latus Rectum:**

$$Latus\ Rectum = \frac{2b^2}{a}$$

Hyperbola

- A **Hyperbola** is the locus of a point whose difference of distances from two fixed points (foci) is always constant.
- **Standard Equation:**

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

- **Eccentricity (e):** For a hyperbola, the eccentricity is always greater than 1 ($e > 1$).
- **Formula for Eccentricity:**

$$e = \sqrt{1 + \frac{b^2}{a^2}}$$

- **Coordinates of the Foci:**

$$(\pm ae, 0)$$

- **Length of Latus Rectum:**

$$Latus\ Rectum\ = \frac{2b^2}{a}$$

TOP 5 QUESTIONS

Q1. Find the coordinates of the focus and the length of the latus rectum of the parabola $y^2 = 16x$.

Answer-

Comparing $y^2 = 16x$ with the standard equation $y^2 = 4ax$:

$$4a = 16 \implies a = 4$$

The focus is $(a, 0)$ and the length of the latus rectum is $4a$.

Focus = (4, 0) and Latus Rectum = 16 units.

Q2. Find the eccentricity of the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$.

Answer-

Comparing with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we get $a^2 = 25$ and $b^2 = 9$.

Using the eccentricity formula:

$$e = \sqrt{1 - \frac{b^2}{a^2}}$$

$$e = \sqrt{1 - \frac{9}{25}} = \sqrt{\frac{25 - 9}{25}}$$

$$e = \sqrt{\frac{16}{25}} = \frac{4}{5}$$

Q3. Find the equation of the hyperbola with vertices $(\pm 3, 0)$ and foci $(\pm 5, 0)$.

Answer-

Here, vertices are $(\pm a, 0) \implies a = 3$, so $a^2 = 9$.

Foci are $(\pm c, 0) \implies c = 5$.

For a hyperbola, $c^2 = a^2 + b^2$:

$$25 = 9 + b^2 \implies b^2 = 16$$

Using standard equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$:

$$\frac{x^2}{9} - \frac{y^2}{16} = 1$$

Q4. Find the equation of a parabola with focus $(2, 0)$ and directrix $x = -2$.

Answer-

Since the focus is $(a, 0) = (2, 0)$ and directrix is $x = -a = -2$, the parabola is of the form $y^2 = 4ax$.

Here, $a = 2$.

Substituting the value of a:

$$y^2 = 4(2)x$$

$y^2 = 8x$

Q5. Find the length of the latus rectum of the ellipse $4x^2 + 9y^2 = 36$.

Answer-

Divide the entire equation by 36 to convert it into standard form:

$$\frac{4x^2}{36} + \frac{9y^2}{36} = 1$$

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

Here, $a^2 = 9 \implies a = 3$, and $b^2 = 4$.

Length of Latus Rectum = $\frac{2b^2}{a}$:

$$\frac{2(4)}{3}$$

$$\Rightarrow \frac{8}{3}$$

5

Matrices

Introduction

In the middle of the 19th Century, **Arthur Cayley** created a new discipline of mathematics called **matrices** to represent simultaneous systems of equations. Today, matrices are widely used in game theory, economics, budgeting, and solving complex linear equations efficiently across various scientific fields.

Matrix and its Order

- A **Matrix** is an ordered rectangular array of numbers or functions enclosed in brackets [] or (). The numbers inside are called elements.
- The horizontal lines of elements are **Rows**, and the vertical lines are **Columns**.
- If a matrix has m rows and n columns, its **Order** is written as $m \times n$ (read as 'm by n').
- **Example:**

$$A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \\ 0 & 4 \end{bmatrix}$$

This is a matrix of order 3×2 because it has 3 rows and 2 columns.

Types of Matrices

1. Row Matrix: A matrix having exactly one row.

- **Example:** $A = \begin{bmatrix} 1 & 5 & 9 \end{bmatrix}$ (Order 1×3)

2. Column Matrix: A matrix having exactly one column.

- **Example:** $B = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$ (Order 3×1)

3. Square Matrix: A matrix where the number of rows equals the number of columns ($m = n$).

- **Example:** $C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ (Order 2×2)

4. Rectangular Matrix: A matrix where the number of rows is not equal to the columns ($m \neq n$).

- **Example:** $D = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ (Order 2×3)

5. Diagonal Matrix: A square matrix where all non-diagonal elements are exactly zero.

- **Example:** $E = \begin{bmatrix} 5 & 0 \\ 0 & 9 \end{bmatrix}$

6. Scalar Matrix: A diagonal matrix where all the diagonal elements are exactly the same.

- **Example:** $F = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$

7. Unit (Identity) Matrix: A scalar matrix where all diagonal elements are exactly 1. Denoted by I.

- **Example:** $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

8. Zero (Null) Matrix: A matrix where every single element is zero. Denoted by O.

- **Example:** $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Equality of Two Matrices

Two matrices A and B are **Equal** (A = B) only if they have the exact same order and their corresponding elements are exactly equal $(a_{ij} = b_{ij})$.

- **Example:** If $\begin{bmatrix} x & 2 \\ 3 & y \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 3 & 7 \end{bmatrix}$, then $x = 5$ and $y = 7$.

Addition and Subtraction of Matrices

- Matrices can be added or subtracted only if they have the **exact same order**.
- **Addition:** Add the corresponding elements. $(C_{ij} = a_{ij} + b_{ij})$
- **Subtraction:** Subtract the corresponding elements. $(C_{ij} = a_{ij} - b_{ij})$
- **Properties of Addition:** Matrix addition is commutative $(A + B = B + A)$ and associative $((A + B) + C = A + (B + C))$.

- **Example:**

$$\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}+\begin{bmatrix}5 & 6\\7 & 8\end{bmatrix}=\begin{bmatrix}1+5 & 2+6\\3+7 & 4+8\end{bmatrix}=\begin{bmatrix}6 & 8\\10 & 12\end{bmatrix}$$

Multiplication of a Matrix by a Scalar

When a matrix is multiplied by a scalar (a constant number) k, every single element inside the matrix is multiplied by k.

- **Formula:**

$$k\begin{bmatrix}a & b\\c & d\end{bmatrix}=\begin{bmatrix}ka & kb\\kc & kd\end{bmatrix}$$

- **Example:**

$$3\begin{bmatrix}2 & -1\\0 & 4\end{bmatrix}=\begin{bmatrix}6 & -3\\0 & 12\end{bmatrix}$$

Multiplication of Two Matrices

- Two matrices A and B can be multiplied to form AB only if the **number of columns in A equals the number of rows in B**.
- If matrix A has order $m \times n$ and matrix B has order $n \times p$, the resulting matrix AB will have an order of $m \times p$.
- We multiply the rows of the first matrix by the corresponding columns of the second matrix and add the products.
- **Properties:** Matrix multiplication is generally **not commutative** $(AB \neq BA)$ but it is associative $((AB)C = A(BC))$.
- **Example:**

$$\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}(1\times 2)+(2\times 1)\\(3\times 2)+(4\times 1)\end{bmatrix}=\begin{bmatrix}4\\10\end{bmatrix}$$

Transpose of a Matrix

- The **Transpose** of a matrix A is obtained by interchanging its rows and columns. It is denoted by A' or A^T.
- If the order of matrix A is $m \times n$, then the order of A' becomes $n \times m$.
- **Properties:** $(A')' = A, (A+B)' = A' + B', and\ (AB)' = B'A'$.

- **Example:** If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$, then $A' = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$.

Symmetric and Skew-Symmetric Matrices

Symmetric Matrix:

A square matrix A is symmetric if it is exactly equal to its transpose. ($A = A'$). The elements satisfy $a_{ij} = a_{ji}$.

- **Example:** $A = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$ (Notice $A = A'$)

Skew-Symmetric Matrix:

A square matrix A is skew-symmetric if it is equal to the negative of its transpose. ($A = -A'$). All its principal diagonal elements are zero, and $a_{ij} = -a_{ji}$.

- **Example:** $B = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$

Theorem: Any square matrix can be expressed as the sum of a symmetric and a skew-symmetric matrix using the formula:

$$A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A').$$

TOP 5 QUESTIONS

Q1. Construct a 2×2 matrix $A = [a_{ij}]$ whose elements are given by $a_{ij} = 2i - j$.

Answer-

Substitute i and j values (1 and 2):

$$a_{11} = 2(1) - 1 = 1$$

$$a_{12} = 2(1) - 2 = 0$$

$$a_{21} = 2(2) - 1 = 3$$

$$a_{22} = 2(2) - 2 = 2$$

$Matrix\ A = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$

Q2. Find the values of a and b if $\begin{bmatrix} a+b & 2 \\ 5 & ab \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 5 & 8 \end{bmatrix}$.

Answer-

Equating corresponding elements:

$$a + b = 6 \implies b = 6 - a$$

$$ab = 8$$

Substitute b:

$$a(6 - a) = 8$$

$$6a - a^2 = 8 \implies a^2 - 6a + 8 = 0$$

Factorizing: $(a - 4)(a - 2) = 0 \implies a = 4$ or $a = 2$.

If $a = 4, b = 2$. If $a = 2, b = 4$.

a = 4, b = 2 OR a = 2, b = 4

Q3. If $A = \begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$, find $3A - B$.

Answer-

First, find $3A$:

$$3A = \begin{bmatrix} 6 & 12 \\ 9 & 6 \end{bmatrix}$$

Now subtract B:

$$3A - B = \begin{bmatrix} 6 - 1 & 12 - 3 \\ 9 - (-2) & 6 - 5 \end{bmatrix}$$

$$3A - B = \begin{bmatrix} 5 & 9 \\ 11 & 1 \end{bmatrix}$$

Q4. Find the product AB if $A = \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$.

Answer-

A is 2×2 and B is 2×3. The result will be 2×3.

$$c_{11} = (1)(1) + (-2)(2) = 1 - 4 = -3$$

$$c_{12} = (1)(2) + (-2)(3) = 2 - 6 = -4$$

$$c_{13} = (1)(3) + (-2)(1) = 3 - 2 = 1$$

$$c_{21} = (2)(1) + (3)(2) = 2 + 6 = 8$$

$$c_{22} = (2)(2) + (3)(3) = 4 + 9 = 13$$

$$c_{23} = (2)(3) + (3)(1) = 6 + 3 = 9$$

$\Rightarrow \mathbf{AB} = \begin{bmatrix} -3 & -4 & 1 \\ 8 & 13 & 9 \end{bmatrix}$

Q5. Express matrix $A = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$ as the sum of a symmetric and a skew-symmetric matrix.

Answer-

Find A':

$$A' = \begin{bmatrix} 3 & 1 \\ 5 & -1 \end{bmatrix}$$

Symmetric part $P = \frac{1}{2}(A + A')$:

$$P = \frac{1}{2}\begin{bmatrix} 6 & 6 \\ 6 & -2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & -1 \end{bmatrix}$$

Skew-symmetric part $Q = \frac{1}{2}(A - A')$:

$$Q = \frac{1}{2}\begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$$

$\Rightarrow \mathbf{A} = \begin{bmatrix} 3 & 3 \\ 3 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$

6

Determinants

Introduction

Every square matrix is associated with a unique number called the **determinant** of the matrix. In this chapter, we will learn various properties of determinants and also evaluate determinants by different methods. This concept is crucial for solving systems of linear equations and finding the inverse of matrices.

Determinant of Order 2

- The **Value of the Determinant** of a 2 × 2 matrix determines whether the values of x and y exist when solving a system of linear equations.
- The number $a_1b_2 - a_2b_1$ is the value of the determinant.
- **Formula:**

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

Expansion of a Determinant of Order 2

- In the determinant $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$, multiply the elements diagonally.
- The sign of the downward arrow is positive ($a_{11}a_{22}$), and the upward arrow is negative ($-a_{21}a_{12}$).
- **Formula:**

$$|A| = a_{11}a_{22} - a_{21}a_{12}$$

- **Example:**

$$\begin{vmatrix} 6 & 4 \\ 8 & 2 \end{vmatrix} = (6 \times 2) - (8 \times 4) = 12 - 32 = -20$$

Determinant of Order 3

- A **Determinant of Order 3** contains nine quantities in 3 rows and 3 columns.
- It has $(3)^2 = 9$ elements.
- It is usually denoted by $\Delta \; or \; |A|$.

Determinant of a Square Matrix

- With each square matrix of numbers, we associate a **Determinant of the matrix**.
- A square matrix whose determinant is zero is called a **Singular Matrix**.
- The determinant of a unit matrix I is exactly 1.
- **Example:**

$$\text{If } A = \begin{bmatrix} 3 & 6 \\ 1 & 5 \end{bmatrix}, \text{ then } |A| = \begin{vmatrix} 3 & 6 \\ 1 & 5 \end{vmatrix} = 3 \times 5 - 1 \times 6 = 15 - 6 = 9.$$

Expansion of a Determinant of Order 3

- A determinant can be expanded using any row or column.
- The elements are assigned positive and negative signs alternatively, starting with a positive sign for a_{11}.
- If the sum of the subscripts is even, assign a positive sign; if odd, assign a negative sign.
- **Example:**

Expanding $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 3 & 2 & 5 \end{vmatrix}$ using the first row:

$$= 1(20 - 2) - 2(10 - 3) + 3(4 - 12) = 18 - 14 - 24 = -20.$$

Minors and Cofactors

Minor of a_{ij} in |A|:

- The **Minor** of an element is the value of the determinant obtained by deleting the row and column containing the element. It is denoted by M_{ij}.
- **Example:** Minor of a_{21} in $\begin{vmatrix} 1 & 6 & 3 \\ 5 & 2 & 4 \\ 7 & 0 & 8 \end{vmatrix}$ is $M_{21} = \begin{vmatrix} 6 & 3 \\ 0 & 8 \end{vmatrix} = 48 - 0 = 48.$

Cofactors of a_{ij} in |A|:

- The **Cofactor** of an element a_{ij} is the minor of a_{ij} multiplied by $(-1)^{i+j}$. It is denoted by C_{ij}.
- **Formula:**

$$C_{ij} = (-1)^{i+j} M_{ij}$$

- **Example:** Cofactor of a_{21} (from above minor) is $C_{21} = (-1)^{2+1}(48) = -48$.

Properties of Determinants

- **Property 1:** The value of a determinant remains unchanged if its rows and columns are interchanged ($\Delta = \Delta'$).
- **Property 2:** If two rows (or columns) of a determinant are interchanged, then the value of the determinant changes in sign only ($\Delta' = -\Delta$).
- **Property 3:** If any two rows (or columns) of a determinant are identical, then the value of the determinant is zero.
- **Property 4:** If each element of a row (or column) of a determinant is multiplied by the same constant k, then the value of the determinant is multiplied by that constant k.

  Corollary: If any two rows (or columns) of a determinant are proportional, then its value is zero.
- **Property 5:** If each element of a row (or column) is expressed as a sum of two terms, the determinant can be expressed as the sum of two determinants.
- **Property 6:** The value of a determinant does not change if to each element of a row (or column) be added the same multiples of corresponding elements of another row (or column).

Evaluation of a Determinant using Properties

- The purpose of simplification is to make maximum possible zeroes in a row (or column) using properties before expanding.
- **Example:**

  Show that $\begin{vmatrix} 1 & w & w^2 \\ w & w^2 & 1 \\ w^2 & 1 & w \end{vmatrix} = 0.$

  By $C_1 \to C_1 + C_2 + C_3$, C_1 becomes $1 + w + w^2$ which equals 0. Thus the determinant is 0.

Application of Determinants

1. Area of a Triangle:

- The area of a triangle with vertices $(x_1, y_1), (x_2, y_2)$ *and* (x_3, y_3) is calculated using a determinant.
- **Formula:**

$$\text{Area} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

2. Condition of collinearity of three points:

- Three points A, B, C are **Collinear** if the area of the triangle formed by them is exactly 0.

3. Equation of a line passing through the given two points:

- The equation of a line joining points (x_1, y_1) and (x_2, y_2) is given by substituting a general point (x, y) into the area formula and equating to zero.

- **Formula:**

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

TOP 5 QUESTIONS

Q1. Find the value of x if $\begin{vmatrix} x-3 & x \\ x+1 & x+3 \end{vmatrix} = 6$.

Answer-

Expand the determinant:

$$(x-3)(x+3) - x(x+1) = 6$$

$$(x^2 - 9) - (x^2 + x) = 6$$

$$-x - 9 = 6$$

x = -15

Q2. Find the minor and cofactor of the element 5 in the determinant $\begin{vmatrix} 1 & 6 & 3 \\ 5 & 2 & 4 \\ 7 & 0 & 8 \end{vmatrix}$.

Answer-

The element 5 is at position a_{21}.

Minor $M_{21} = \begin{vmatrix} 6 & 3 \\ 0 & 8 \end{vmatrix} = (6 \times 8) - (3 \times 0) = 48.$

Cofactor $C_{21} = (-1)^{2+1} \times 48 = -48.$

Minor = 48, Cofactor = -48.

Q3. Evaluate the determinant $\begin{vmatrix}1 & 2 & 3\\ 2 & 4 & 1\\ 3 & 2 & 5\end{vmatrix}$ by expanding along the first row.

Answer-

Using the first row expansion:

$$= 1|^{4 \quad 1}_{2 \quad 5}| - 2|^{2 \quad 1}_{3 \quad 5}| + 3|^{2 \quad 4}_{3 \quad 2}|$$

$$= 1(20 - 2) - 2(10 - 3) + 3(4 - 12)$$

$$= 18 - 14 - 24$$

The value is -20.

Q4. Find the area of the triangle with vertices $P(5,4), Q(-2,4)$ and $R(2,-6)$.

Answer-

Using the determinant formula for area:

$$\text{Area} = \frac{1}{2}|\begin{matrix}5 & 4 & 1\\ -2 & 4 & 1\\ 2 & -6 & 1\end{matrix}|$$

$$= \frac{1}{2}[5(4 - (-6)) - 4(-2 - 2) + 1(12 - 8)]$$

$$= \frac{1}{2}[50 + 16 + 4] = \frac{70}{2}$$

Area = 35 sq units.

Q5. Find the equation of the line joining points $(1,3)$ and $(2,1)$ using determinants.

Answer-

Let $P(x, y)$ be any point on the line. Since points are collinear, the area determinant is zero:

$$|\begin{matrix}x & y & 1\\ 1 & 3 & 1\\ 2 & 1 & 1\end{matrix}| = 0$$

$$x(3 - 1) - y(1 - 2) + 1(1 - 6) = 0$$

2x + y - 5 = 0

7

Inverse of a Matrix and its Applications

Introduction

In previous chapters, we learned about matrices, determinants, and their basic operations. In this chapter, we will learn to find the adjoint of a matrix and use it to calculate the **inverse of a square matrix**. Furthermore, we will apply these concepts to solve systems of linear equations and check their consistency.

Adjoint of a Square Matrix

- Let A be a square matrix. The **Adjoint of a Matrix** A is the transpose of the matrix formed by the cofactors of its elements.
- It is denoted by **Adj A**.
- **Formula:**

$$\text{Adj } A = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^T = \begin{bmatrix} C_{11} & C_{21} \\ C_{12} & C_{22} \end{bmatrix}$$

- **Example:**

Find the adjoint of matrix $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$.

Cofactors: $C_{11} = 4, C_{12} = -1, C_{21} = -3, C_{22} = 2.$

Matrix of cofactors = $\begin{bmatrix} 4 & -1 \\ -3 & 2 \end{bmatrix}$.

$\text{Adj } A = \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$.

Inverse of a Matrix

- A square matrix A is called **Invertible** if there exists another square matrix B such that AB = BA = I (Identity Matrix).
- The matrix B is called the **Inverse** of A and is denoted by A^{-1}.
- The inverse of a matrix exists only if it is a **Non-singular Matrix** (i.e., $|A| \neq 0$).
- **Formula:**

$$A^{-1} = \frac{1}{|A|} \text{Adj } A$$

- **Example:**

Find the inverse of $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$.
$|A| = (2 \times 3) - (5 \times 1) = 6 - 5 = 1$. Since $|A| \neq 0$, A^{-1} exists.
$\text{Adj } A = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$.
$A^{-1} = \frac{1}{1}\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$.

Solution of a System of Linear Equations

- A system of linear equations can be written in **Matrix Form** as $AX = B$.
- Here, A is the matrix of coefficients, X is the column matrix of variables, and B is the column matrix of constants.
- The solution to the system is found by multiplying both sides by the inverse of A.
- **Formula:**

$$X = A^{-1}B$$

- **Example:**

Solve: $x + 2y = 4$ and $2x + 5y = 9$.
$A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, $B = \begin{bmatrix} 4 \\ 9 \end{bmatrix}$.
$|A| = 5 - 4 = 1$. $\text{Adj } A = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$.
$A^{-1} = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$.
$X = A^{-1}B = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 9 \end{bmatrix} = \begin{bmatrix} 20 - 18 \\ -8 + 9 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.
Therefore, $x = 2, y = 1$.

Criterion for Consistency of a System of Equations

- A system of equations is **Consistent** if it has one or more valid solutions.
- A system of equations is **Inconsistent** if it has no solution at all.
- **Condition 1 (Unique Solution):** If $|A| \neq 0$, the system is consistent and has a unique solution.
- **Condition 2 (No Solution):** If $|A| = 0$ and $(Adj\ A)\ B \neq O$ (Zero Matrix), the system is inconsistent.
- **Condition 3 (Infinite Solutions):** If $|A| = 0$ and $(Adj\ A)\ B = O$, the system is consistent and has infinitely many solutions.

- **Example:**

Check consistency of $2x+3y=5$ and $4x+6y=10$.

$|A| = \begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} = 12 - 12 = 0.$

$\text{Adj } A = \begin{bmatrix} 6 & -3 \\ -4 & 2 \end{bmatrix}.$

$(\text{Adj } A)B = \begin{bmatrix} 6 & -3 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 10 \end{bmatrix} = \begin{bmatrix} 30-30 \\ -20+20 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$

Since $|A| = 0$ and $(\text{Adj } A)B = O$, the system has infinite solutions (Consistent).

TOP 5 QUESTIONS

Q1. Find the adjoint of the matrix $A = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$.

Answer-

Find the cofactors of A:

$C_{11} = 4, C_{12} = -3$

$C_{21} = 2, C_{22} = 1$

Form the cofactor matrix and take its transpose.

$\text{Adj A} = \begin{bmatrix} 4 & 2 \\ -3 & 1 \end{bmatrix}$

Q2. Find the inverse of the matrix $A = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$.

Answer-

Calculate the determinant:

$$|A| = (2 \times 3) - (-1 \times 1) = 6 + 1 = 7$$

Since $|A| \neq 0$, the inverse exists.

$$\text{Adj } A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{Adj } A$$

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$$

Q3. Solve the system of equations using the matrix method: $5x + 2y = 4$ and $7x + 3y = 5$.

Answer-

Matrix form $AX = B$: $A = \begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$.

$$|A| = 15 - 14 = 1$$

$$\text{Adj } A = \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}$$

$$X = A^{-1}B = \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 12 - 10 \\ -28 + 25 \end{bmatrix}$$

x = 2, y = -3

Q4. Determine if the system is consistent: $x + 2y = 4$ and $2x + 4y = 7$.

Answer-

$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$.

$$|A| = 4 - 4 = 0$$

$$\text{Adj } A = \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$$

$$(\text{Adj } A)B = \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 7 \end{bmatrix} = \begin{bmatrix} 16 - 14 \\ -8 + 7 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Since $(\text{Adj } A)B \neq O$.

The system is inconsistent (no solution).

Q5. If $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ verify that $A\,(Adj\ A) = |A|I$.

Answer-

$|A| = 4 - 3 = 1$. So, $|A|I = 1\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$.

$$\text{Adj } A = \begin{bmatrix}2 & -3\\-1 & 2\end{bmatrix}$$

$$A(\text{Adj } A) = \begin{bmatrix}2 & 3\\1 & 2\end{bmatrix}\begin{bmatrix}2 & -3\\-1 & 2\end{bmatrix}$$

$$= \begin{bmatrix}4-3 & -6+6\\2-2 & -3+4\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

Hence verified, A(Adj A) = |A|I.

8

Introduction to Three-Dimensional Geometry

Introduction

Three-dimensional geometry extends the concepts of two-dimensional coordinate geometry into the real physical world. In this chapter, we will learn how to locate points in space using three axes, calculate distances between them, apply section formulas, and understand the crucial concepts of direction cosines and direction ratios.

Coordinate System and Coordinates of a Point in Space

- Three mutually perpendicular lines intersecting at the **Origin** $O(0,0,0)$ form the rectangular coordinate axes: the x-axis, y-axis, and z-axis.
- These three axes mutually divide the space into eight distinct compartments known as **Octants**.
- The coordinates of any point P in space are uniquely represented by an ordered triplet (x, y, z).
- The coordinate planes are the xy-plane (where z=0), the yz-plane (where x=0), and the zx-plane (where $y = 0$).

Distance Between Two Points

- The distance d between any two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is determined using the 3D distance formula.
- **Formula:**

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

- The distance of a point $P(x, y, z)$ from the **Origin** $(0,0,0)$ is calculated as $\sqrt{x^2 + y^2 + z^2}$.
- **Example:** Find the distance between the points $(2, 5, -4)$ and $(8, 2, -6)$.

$$d = \sqrt{(8-2)^2 + (2-5)^2 + \left(-6 - (-4)\right)^2}$$

$$d = \sqrt{(36 + 9 + 4)} = \sqrt{49} = 7\ units$$

Coordinates of a Point of Division of a Line Segment

1. Internal Division:

The coordinates of a point $R(x, y, z)$ dividing the line segment joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ internally in the ratio $m_1 : m_2$.

- **Formula:**

$$x = \frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \quad y = \frac{m_1y_2 + m_2y_1}{m_1 + m_2}, \quad z = \frac{m_1z_2 + m_2z_1}{m_1 + m_2}$$

2. External Division:

If the point R divides the segment externally, replace the '+' sign with a '-' sign in the internal division formula.

- **Formula:**

$$x = \frac{m_1x_2 - m_2x_1}{m_1 - m_2}, \quad y = \frac{m_1y_2 - m_2y_1}{m_1 - m_2}, \quad z = \frac{m_1z_2 - m_2z_1}{m_1 - m_2}$$

3. Mid-point Formula:

When the point divides the line segment exactly in half ($m_1 = m_2 = 1$).

- **Formula:**

$$x = \frac{x_1 + x_2}{2}, \quad y = \frac{y_1 + y_2}{2}, \quad z = \frac{z_1 + z_2}{2}$$

Direction Cosines and Direction Ratios of a Line

1. Direction Cosines (l, m, n):

The cosines of the angles α, β, γ that a directed line makes with the positive directions of the x, y, and z axes respectively.

- **Formulas:**

$$l = \cos\alpha, \quad m = \cos\beta, \quad n = \cos\gamma$$

- **Fundamental Relation:** The sum of the squares of the direction cosines of any line is always exactly 1.

$$l^2 + m^2 + n^2 = 1$$

2. Direction Ratios (a, b, c):

Any three numbers that are proportional to the direction cosines l, m, n of a line.

- **Relation with Direction Cosines:**

$$l = \pm\frac{a}{\sqrt{a^2+b^2+c^2}}, \quad m = \pm\frac{b}{\sqrt{a^2+b^2+c^2}}, \quad n = \pm\frac{c}{\sqrt{a^2+b^2+c^2}}$$

- **Example:** Find the direction cosines of a line whose direction ratios are $2, -1, -2$.

  Here $a = 2, b = -1, c = -2$.

$$\sqrt{a^2+b^2+c^2} = \sqrt{4+1+4} = \sqrt{9} = 3$$

  Direction cosines are $\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}$.

Projection of a Line Segment on a Line

- The **Projection** of a line segment joining two points $P(x_1, y_1, z_1) and\ Q(x_2, y_2, z_2)$ on a given straight line with direction cosines l, m, n.

- **Formula:**

$$Projection = |l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)|$$

TOP 5 QUESTIONS

Q1. Find the distance between the points $A(2, 3, 5)\ and\ B(4, 3, 1)$.

Answer-

Using the 3D Distance Formula:

$$AB = \sqrt{(4-2)^2 + (3-3)^2 + (1-5)^2}$$

$$AB = \sqrt{2^2 + 0^2 + (-4)^2}$$

$$AB = \sqrt{4+0+16} = \sqrt{20}$$

Distance = $2\sqrt{5}\ units$.

Q2. Find the coordinates of the mid-point of the line segment joining $(1, -2, 3)$ and $(3, 4, -1)$.

Answer-

Using the Mid-point Formula:

$$x = \frac{1+3}{2} = \frac{4}{2} = 2$$

$$y = \frac{-2+4}{2} = \frac{2}{2} = 1$$

$$z = \frac{3+(-1)}{2} = \frac{2}{2} = 1$$

The mid-point is (2, 1, 1).

Q3. Find the direction cosines of a line if it makes angles $90°, 60°$ and $30°$ with the x, y and z axes respectively.

Answer-

Here the angles are $\alpha = 90°, \beta = 60°, \gamma = 30°$.

$$l = \cos 90° = 0$$

$$m = \cos 60° = \frac{1}{2}$$

$$n = \cos 30° = \frac{\sqrt{3}}{2}$$

Directions cosines are $\left(0, \frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

Q4. Find the coordinates of the point which divides the line joining $(1, -2, 3)$ and $(3, 4, -5)$ internally in the ratio $2:3$.

Answer-

Using the Internal Section Formula $(m_1 = 2, m_2 = 3)$:

$$x = \frac{2(3)+3(1)}{2+3} = \frac{6+3}{5} = \frac{9}{5}$$

$$y = \frac{2(4)+3(-2)}{2+3} = \frac{8-6}{5} = \frac{2}{5}$$

$$z = \frac{2(-5)+3(3)}{2+3} = \frac{-10+9}{5} = -\frac{1}{5}$$

The point is $\left(\frac{9}{5}, \frac{2}{5}, -\frac{1}{5}\right)$.

Q5. Find the direction cosines of a line whose direction ratios are $1, 2, 3$.

Answer-

Here $a = 1, b = 2, c = 3$.

Find $\sqrt{a^2+b^2+c^2}$:

$$= \sqrt{1^2+2^2+3^2} = \sqrt{1+4+9} = \sqrt{14}$$

Divide each direction ratio by $\sqrt{14}$:

Direction cosines are $\left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right)$.

9

Vectors

Introduction

In day-to-day life, physical quantities like distance and speed describe changes in position. However, to describe displacement or velocity accurately, we must specify direction alongside **magnitude**. This chapter introduces **scalars, vectors,** their types, operations like addition, section formulas, direction cosines, and vector products to predict positions accurately.

Scalars and Vectors

1. A **Scalar** is a physical quantity which can be represented by a number only (magnitude only).

- Examples: Time, mass, length, speed, temperature, volume, quantity of heat, work done.

2. A **Vector** is a physical quantity which has magnitude as well as direction.

- Examples: Displacement, velocity, acceleration, force, weight.

Vector as a Directed Line Segment

- A line segment along with a specific direction is called a **directed line segment**.
- The point A from where the vector starts is called the **initial point** and point B where it ends is called the **terminal point**.
- The length of the line segment AB is called the **magnitude** or modulus of the vector, denoted by $|\overrightarrow{AB}|$ or $\vec{a}$.

Classification of Vectors

- **Zero Vector (Null Vector):** A vector whose magnitude is exactly zero and has no definite direction. It is denoted by $\vec{0}$.
- **Unit Vector:** A vector whose magnitude is exactly unity ($|\vec{a}| = 1$). A unit vector in the direction of $\vec{a}$ is denoted by $\hat{a}$.
- **Equal Vectors:** Two vectors are equal if they have the exact same magnitude and the exact same direction.

- **Like Vectors:** Vectors having the same direction whatever be their magnitudes are called like vectors.
- **Negative of a Vector:** A vector having the same magnitude as a given vector but opposite direction. Denoted by $-\vec{a}$.
- **Co-initial Vectors:** Two or more vectors having the exact same initial point are called co-initial vectors.
- **Collinear Vectors:** Two or more vectors are collinear if they are parallel to the same line whatever be their magnitudes.
- **Co-planar Vectors:** Vectors are called co-planar if they are parallel to the exact same plane.

Addition of Vectors

- **Triangle Law of Addition of Vectors:** If two vectors are represented by two sides of a triangle in order, their sum (resultant) is represented by the third side taken in reverse order.
- **Addition of more than two Vectors:** We use the polygon law of vector addition to find the resultant of multiple vectors step by step.
- **Parallelogram Law of Addition of Vectors:** If two vectors are represented by two adjacent sides of a parallelogram, their resultant is represented by the diagonal passing through the common point.
- **Negative of a Vector:** For any vector $\vec{a}$, the sum of the vector and its negative is the zero vector: $\vec{a} + (-\vec{a}) = \vec{0}$.
- **The Difference of Two Given Vectors:** The difference $\vec{a} - \vec{b}$ is defined as the addition of $\vec{a}$ and the negative of $\vec{b}$: $\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$.

Position Vector of a Point

- The **Position Vector** of a point P with respect to an origin of reference O is the vector $\overrightarrow{OP}$.
- For any two points A and B, the vector $\overrightarrow{AB} = (Position\ vector\ of\ B) - (Position\ vector\ of\ A)$.

Multiplication of a Vector by a Scalar

- The product of a vector $\vec{a}$ by a scalar x is a vector whose magnitude is $|x||\vec{a}|$.
- Its direction is the same as $\vec{a}$ if $x > 0$, and exactly opposite to $\vec{a}$ if $x < 0$.

- **Condition of Collinearity:** Two vectors $\vec{a}$ and $\vec{b}$ are collinear if there exist scalars x and y such that $x\vec{a} + y\vec{b} = \vec{0}$.

Co-planarity of Vectors

- Any vector $\vec{c}$, which is coplanar with two non-collinear vectors $\vec{a}$ *and* $\vec{b}$, is expressible as a linear combination of $\vec{a}$ *and* $\vec{b}$.
- **Formula:** $\vec{c} = l\vec{a} + m\vec{b}$ (where l and m are scalars).

Resolution of a Vector along Two Perpendicular Axes

- In a 2D plane, any vector $\vec{r}$ can be uniquely expressed as the sum of its components along the x-axis and y-axis.
- **Formula:** $\vec{r} = x\hat{\imath} + y\hat{\jmath}$

Resolution of a Vector in Three Dimensions along Three Mutually Perpendicular Axes

- In a 3D space, a vector $\vec{r}$ can be resolved along the $x, y,$ *and* z axes using unit vectors $\hat{\imath}, \hat{\jmath}, \hat{k}$.
- **Formula:** $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$
- **Magnitude:** $|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$

Section Formula

- Let points A and B have position vectors $\vec{a}$ *and* $\vec{b}$. Let point P (position vector $\vec{r}$) divide AB in the ratio $m:n$.
- **Internal Division:** $\vec{r} = \frac{m\vec{b} + n\vec{a}}{m + n}$
- **Mid-point:** $\vec{r} = \frac{\vec{a} + \vec{b}}{2}$
- **External Division:** $\vec{r} = \frac{m\vec{b} - n\vec{a}}{m - n}$

Direction Cosines of a Vector

- The cosines of the angles α, β, γ subtended by a vector with the positive directions of the x, y, and z axes are called its **Direction Cosines**.
- $l = \cos\alpha,\ m = \cos\beta,\ n = \cos\gamma$

- **Relation:** $l^2 + m^2 + n^2 = 1$

Direction Cosines of a Vector joining two points:

For points $(x_1, y_1, z_1)$and (x_2, y_2, z_2), direction cosines are $\frac{x_2-x_1}{r}, \frac{y_2-y_1}{r}, \frac{z_2-z_1}{r}$ where r is the distance between them.

Direction Ratios of a Vector:

Any three real numbers which are proportional to the direction cosines of a vector are called its **Direction Ratios**.

Scalar (or Dot) Product of Two Vectors

- The **Scalar Product** of two non-zero vectors $\vec{a}$ and $\vec{b}$ is defined as the product of their magnitudes and the cosine of the angle θ between them.
- It always yields a scalar quantity (a real number).
- **Formula:** $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|cos\theta$
- **Condition of Perpendicularity:** Two non-zero vectors are perpendicular if and only if their scalar product is exactly zero ($\vec{a} \cdot \vec{b} = 0$).
- **Properties:** $\hat{\imath} \cdot \hat{\imath} = \hat{\jmath} \cdot \hat{\jmath} = \hat{k} \cdot \hat{k} = 1$ and $\hat{\imath} \cdot \hat{\jmath} = \hat{\jmath} \cdot \hat{k} = \hat{k} \cdot \hat{\imath} = 0$.

Vector (or Cross) Product of Two Vectors

- The **Vector Product** of two non-zero vectors $\vec{a}$ and $\vec{b}$ is a vector whose magnitude is the area of the parallelogram formed by them, and its direction is perpendicular to both vectors.
- **Formula:** $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\ sin\theta\ \hat{n}$ (where $\hat{n}$ is a unit vector perpendicular to both).
- **Condition of Parallelism:** Two non-zero vectors are parallel or collinear if their vector product is the zero vector ($\vec{a} \times \vec{b} = \vec{0}$).
- **Properties:** $\hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k} = \vec{0}$ and $\hat{\imath} \times \hat{\jmath} = \hat{k},\ \hat{\jmath} \times \hat{k} = \hat{\imath},\ \hat{k} \times \hat{\imath} = \hat{\jmath}$.

Scalar Triple Product of Vectors

- The **Scalar Triple Product** of three vectors $\vec{a}, \vec{b}, \vec{c}$ is the scalar product of one vector with the vector product of the other two.

- It represents the volume of a parallelepiped formed by the three vectors as its adjacent edges.
- **Formula:** $[\vec{a}\,\vec{b}\,\vec{c}] = \vec{a} \cdot (\vec{b} \times \vec{c})$
- **Condition of Coplanarity:** Three non-zero vectors are strictly **Co-planar** if their scalar triple product is exactly zero ($[\vec{a}\,\vec{b}\,\vec{c}] = 0$).

TOP 5 QUESTIONS

Q1. Find the magnitude of the vector $\vec{a} = 3\hat{\imath} - 2\hat{\jmath} + 6\hat{k}$.

Answer-

The magnitude of a vector is given by:

$$|\vec{a}| = \sqrt{x^2 + y^2 + z^2}$$

Substituting the components:

$$|\vec{a}| = \sqrt{(3)^2 + (-2)^2 + (6)^2}$$
$$|\vec{a}| = \sqrt{9 + 4 + 36}$$
$$|\vec{a}| = \sqrt{49} = 7$$

Final Answer: $|\vec{a}| = 7$ units

Q2. Find the scalar (dot) product of the vectors $\vec{a} = 2\hat{\imath} + 3\hat{\jmath} - \hat{k}, \vec{b} = \hat{\imath} - 2\hat{\jmath} + 3\hat{k}$.

Answer-

The dot product is:

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$
$$\vec{a} \cdot \vec{b} = (2)(1) + (3)(-2) + (-1)(3)$$
$$= 2 - 6 - 3$$
$$= -7$$

Final Answer: $\vec{a} \cdot \vec{b} = -7$

Q3. Find the angle between the vectors $\vec{a} = \hat{\imath} + \hat{\jmath} + \hat{k}, \vec{b} = \hat{\imath} + \hat{\jmath} - \hat{k}$.

Answer-

First, magnitudes:

$$|\vec{a}| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$
$$|\vec{b}| = \sqrt{1^2 + 1^2 + (-1)^2} = \sqrt{3}$$

Dot product:

$$\vec{a} \cdot \vec{b} = (1)(1) + (1)(1) + (1)(-1)$$
$$= 1 + 1 - 1 = 1$$

Using formula:

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$$
$$\cos\theta = \frac{1}{\sqrt{3} \cdot \sqrt{3}} = \frac{1}{3}$$
$$\theta = \cos^{-1}\left(\frac{1}{3}\right)$$

Final Answer: $\theta = \cos^{-1}\left(\frac{1}{3}\right) \approx 70.53^\circ$

Q4. Find the vector (cross) product $\vec{a} = 2\hat{\imath} + \hat{\jmath} - \hat{k}, \vec{b} = \hat{\imath} - \hat{\jmath} + 2\hat{k}$.

Answer-

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 1 & -1 \\ 1 & -1 & 2 \end{vmatrix}$$

Expanding:

$$= \hat{\imath}(1 \cdot 2 - (-1)(-1)) - \hat{\jmath}(2 \cdot 2 - (-1)(1)) + \hat{k}(2(-1) - 1 \cdot 1)$$
$$= \hat{\imath}(2 - 1) - \hat{\jmath}(4 - (-1)) + \hat{k}(-2 - 1)$$
$$= \hat{\imath}(1) - \hat{\jmath}(5) + \hat{k}(-3)$$
$$= \hat{\imath} - 5\hat{\jmath} - 3\hat{k}$$

Final Answer: $\vec{a} \times \vec{b} = \hat{\imath} - 5\hat{\jmath} - 3\hat{k}$

Q5. Check whether the vectors $\vec{a} = \hat{\imath} - 2\hat{\jmath} + 3\hat{k}, \vec{b} = -2\hat{\imath} + 3\hat{\jmath} - 4\hat{k}, \vec{c} = \hat{\imath} - 3\hat{\jmath} + 5\hat{k}$ are coplanar.

Answer-

Vectors are coplanar if their scalar triple product is zero:

$$[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} 1 & -2 & 3 \\ -2 & 3 & -4 \\ 1 & -3 & 5 \end{vmatrix}$$

Expanding:

$$= 1(3 \cdot 5 - (-4)(-3)) - (-2)((-2 \cdot 5) - (-4 \cdot 1)) + 3((-2 \cdot -3) - (3 \cdot 1))$$
$$= 1(15 - 12) - (-2)(-10 - (-4)) + 3(6 - 3)$$
$$= 1(3) - (-2)(-6) + 3(3)$$
$$= 3 - 12 + 9 = 0$$

Final Answer: $[\vec{a}\ \vec{b}\ \vec{c}] = 0 \Rightarrow$ Vectors are coplanar.

10

Plane

Introduction

A plane is a two-dimensional flat surface that extends infinitely in three-dimensional space. In this chapter, we will learn how to derive the algebraic and vector equations of a **plane** under various conditions, such as passing through specific points, having specific intercepts, or finding distances and angles between them.

Equation of a Plane in Normal Form

- A **Plane** is uniquely determined if the perpendicular distance from the origin and the direction of the normal are known.
- **Vector Form:** The equation of a plane at a distance d from the origin with unit normal vector $\hat{n}$.

$$\vec{r} \cdot \hat{n} = d$$

- **Cartesian Form:** If l, m, n are the direction cosines of the normal, the equation is:

$$lx + my + nz = d$$

Equation of a Plane passing through a given point and perpendicular to a given vector

- A plane is completely determined if it passes through a known point and is perpendicular to a known normal vector.
- **Vector Form:** Where $\vec{a}$ is the position vector of the point and $\vec{n}$ is the normal vector.

$$(\vec{r} - \vec{a}) \cdot \vec{n} = 0$$

- **Cartesian Form:** Where (x_1, y_1, z_1) is the given point and A, B, C are direction ratios of the normal.

$$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$$

Equation of a Plane passing through Three Non-Collinear Points

- A unique plane always passes through three given non-collinear points.
- **Vector Form:** Where $\vec{a}, \vec{b}, \vec{c}$ are position vectors of the three points.

$$(\vec{r} - \vec{a}) \cdot [(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})] = 0$$

- **Cartesian Form:**

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

Equation of a Plane in the Intercept Form

- The **Intercept Form** represents the equation of a plane making specific intercepts on the x, y, and z coordinate axes.
- **Formula:** Where a, b, c are the x, y, and z intercepts respectively.

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

Equation of a Plane passing through the intersection of two given planes

- The infinite family of planes passing through the line of intersection of two given planes.
- **Vector Form:**

$$(\vec{r} \cdot \vec{n_1} - d_1) + \lambda (\vec{r} \cdot \vec{n_2} - d_2) = 0$$

- **Cartesian Form:**

$$(A_1x + B_1y + C_1z - D_1) + \lambda (A_2x + B_2y + C_2z - D_2) = 0$$

Angle between Two Planes

- The **Angle between two planes** is defined exactly as the angle between their respective normals.
- **Vector Form:**

$$cos\theta = \left| \frac{\vec{n_1} \cdot \vec{n_2}}{|\vec{n_1}|\,|\vec{n_2}|} \right|$$

- **Cartesian Form:**

$$\cos\theta = \left| \frac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2}\sqrt{A_2^2 + B_2^2 + C_2^2}} \right|$$

- **Condition of Perpendicularity:** $\vec{n_1} \cdot \vec{n_2} = 0$

- **Condition of Parallelism:** $\vec{n_1} \times \vec{n_2} = \vec{0}$

Distance of a Point from a Plane

- The perpendicular distance of a specific point from a given plane.

- **Vector Form:** Distance from a point with position vector $\vec{a}$ to the plane $\vec{r} \cdot \hat{n} = d$.

$$Distance = |\vec{a} \cdot \hat{n} - d|$$

- **Cartesian Form:** Distance from (x_1, y_1, z_1) to Ax + By + Cz + D = 0.

$$d = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

TOP 5 QUESTIONS

Q1. Find the vector equation of a plane which is at a distance of 7units from the origin and normal to the vector $\vec{n} = 3\hat{\imath} + 5\hat{\jmath} - 6\hat{k}$.

Answer-

First, find the magnitude of the normal vector:

$$|\vec{n}| = \sqrt{3^2 + 5^2 + (-6)^2} = \sqrt{9 + 25 + 36} = \sqrt{70}$$

Unit normal vector:

$$\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{3\hat{\imath} + 5\hat{\jmath} - 6\hat{k}}{\sqrt{70}}$$

Using the normal form of plane:

$$\vec{r} \cdot \hat{n} = d$$

$$\vec{r} \cdot \left(\frac{3}{\sqrt{70}}\hat{\imath} + \frac{5}{\sqrt{70}}\hat{\jmath} - \frac{6}{\sqrt{70}}\hat{k}\right) = 7$$

Multiply both sides by $\sqrt{70}$ to simplify:

$$\vec{r} \cdot (3\hat{\imath} + 5\hat{\jmath} - 6\hat{k}) = 7\sqrt{70}$$

Final Answer: $\vec{r} \cdot (3\hat{\imath} + 5\hat{\jmath} - 6\hat{k}) = 7\sqrt{70}$

Q2. Find the Cartesian equation of the plane passing through the point $(1, 2, -4)$and perpendicular to direction ratios $2, 3, -1$.

Answer-

The normal vector is:

$$\vec{n} = 2\hat{\imath} + 3\hat{\jmath} - \hat{k}$$

Using plane equation:

$$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$$

Substitute values:

$$2(x - 1) + 3(y - 2) - 1(z - (-4)) = 0$$
$$2x - 2 + 3y - 6 - z - 4 = 0$$
$$2x + 3y - z - 12 = 0$$

Final Answer: $2x + 3y - z - 12 = 0$

Q3. Find the intercepts cut off by the plane $2x + y - z = 5$.

Answer-

Divide the equation by 5:

$$\frac{2x}{5} + \frac{y}{5} - \frac{z}{5} = 1$$

Convert into intercept form:

$$\frac{x}{5/2} + \frac{y}{5} + \frac{z}{-5} = 1$$

So intercepts are:

$$x = \frac{5}{2}, y = 5, z = -5$$

Final Answer: x-intercept $= \frac{5}{2}$, y-intercept $= 5$, z-intercept $= -5$.

Q4. Find the angle between the planes $2x + y - 2z = 5$ and $3x - 6y - 2z = 7$.

Answer-

Normal vectors:

$$\vec{n}_1 = 2\hat{\imath} + \hat{\jmath} - 2\hat{k}, \vec{n}_2 = 3\hat{\imath} - 6\hat{\jmath} - 2\hat{k}$$

Dot product:

$$\vec{n}_1 \cdot \vec{n}_2 = (2)(3) + (1)(-6) + (-2)(-2) = 6 - 6 + 4 = 4$$

Magnitudes:

$$|\vec{n}_1| = \sqrt{4 + 1 + 4} = 3, |\vec{n}_2| = \sqrt{9 + 36 + 4} = 7$$

$$\cos\theta = \frac{4}{3 \times 7} = \frac{4}{21}$$

$$\theta = \cos^{-1}\left(\frac{4}{21}\right)$$

$$\theta \approx 79.0^\circ$$

Final Answer: $\theta = \cos^{-1}\left(\frac{4}{21}\right) \approx 79.0^\circ$

Q5. Find the distance of the point $(2, 5, -3)$ from the plane $\vec{r} \cdot (6\hat{\imath} - 3\hat{\jmath} + 2\hat{k}) = 4$.

Answer-

Convert into Cartesian form:

$$6x - 3y + 2z - 4 = 0$$

Using distance formula:

$$d = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

$$d = \frac{|6(2) - 3(5) + 2(-3) - 4|}{\sqrt{6^2 + (-3)^2 + 2^2}}$$

$$= \frac{|\,12 - 15 - 6 - 4\,|}{\sqrt{36 + 9 + 4}} = \frac{|\,-13\,|}{\sqrt{49}}$$

$$d = \frac{13}{7}$$

Final Answer: $d = \frac{13}{7}$ units

11

Straight Line

Introduction

A straight line in three-dimensional space is uniquely determined if it passes through a known point in a specified direction, or if it passes through two distinct points. This chapter explains the vector and Cartesian equations of lines, the angle between them, and the shortest distance between skew lines.

Vector equation of a line

1. Vector equation of a line passing through a given point and parallel to a given vector

- Let a line pass through a given point A with position vector $\vec{a}$ and be parallel to a specific given vector $\vec{m}$.
- Let $\vec{r}$ be the position vector of any arbitrary point P on this straight line.
- **Formula:**

$$\vec{r} = \vec{a} + \lambda \vec{m}$$

(where λ is a scalar parameter)

2. Cartesian equation of a line passing through a given point and having given direction cosines

- Let a line pass through a fixed point $(x_1, y_1, z_1)$and have given **direction cosines** l, m, n.
- The coordinates (x, y, z) of any point on the line satisfy a proportional relationship with the direction cosines.
- **Formula:**

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

- If the **direction ratios** a, b, c are given instead of direction cosines, the formula remains similar.
- **Formula with Direction Ratios:**

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

3. Equation of a line passing through two given points

- A unique straight line always passes exactly through two given distinct points A and B.

- **Vector Form:** Let $\vec{a}$ *and* $\vec{b}$ be the position vectors of the two points A and B. The direction vector becomes $(\vec{b} - \vec{a})$.

$$\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$$

- **Cartesian Form:** Let the two points be (x_1, y_1, z_1) and (x_2, y_2, z_2). The direction ratios are the differences of their coordinates.

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

Angle between two lines

- The **Angle** θ between two straight lines is defined exactly as the angle between their respective direction vectors $\overrightarrow{m_1}$ *and* $\overrightarrow{m_2}$.

- **Vector Form:**

$$\cos\theta = \frac{m_1 \cdot m_2}{|m_1||m_2|}$$

- **Cartesian Form:** If the lines have direction ratios a_1, b_1, c_1 and a_2, b_2, c_2.

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

- **Condition for Perpendicularity:** Two lines are exactly perpendicular if the dot product of their direction vectors is zero.

$$m_1 \cdot m_2 = 0 \quad \text{or} \quad a_1a_2 + b_1b_2 + c_1c_2 = 0$$

- **Condition for Parallelism:** Two lines are parallel if their direction ratios are strictly proportional.

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

Shortest distance between two lines

- **Skew Lines:** Straight lines in space which are neither parallel nor intersecting are called skew lines. They lie in completely different planes.
- **Shortest Distance (Skew Lines):** The shortest distance d is the perpendicular distance between two skew lines $\vec{r} = \vec{a_1} + \lambda \vec{m_1}$ *and* $\vec{r} = \vec{a_2} + \mu \vec{m_2}$.
- **Formula for Skew Lines:**

$$d = \left| \frac{(a_2 - a_1) \cdot (m_1 \times m_2)}{|m_1 \times m_2|} \right|$$

- **Condition for Intersecting Lines:** Two lines intersect if and only if the shortest distance between them is exactly zero.

$$(\vec{a_2} - \vec{a_1}) \cdot (\vec{m_1} \times \vec{m_2}) = 0$$

- **Shortest Distance (Parallel Lines):** The perpendicular distance d between two parallel lines $\vec{r} = \vec{a_1} + \lambda \vec{m}$ *and* $\vec{r} = \vec{a_2} + \mu \vec{m}$.
- **Formula for Parallel Lines:**

$$d = \frac{|(a_2 - a_1) \times m|}{|m|}$$

TOP 5 QUESTIONS

Q1. Find the vector equation of the line passing through the point $(5, 2, -4)$ and parallel to the vector $3\hat{\imath} + 2\hat{\jmath} - 8\hat{k}$.

Answer-

Position vector of the given point:

$$\vec{a} = 5\hat{\imath} + 2\hat{\jmath} - 4\hat{k}$$

Direction vector:

$$\vec{m} = 3\hat{\imath} + 2\hat{\jmath} - 8\hat{k}$$

Using the vector equation of a line:

$$\vec{r} = \vec{a} + \lambda\vec{m}$$
$$\vec{r} = (5\hat{\imath} + 2\hat{\jmath} - 4\hat{k}) + \lambda(3\hat{\imath} + 2\hat{\jmath} - 8\hat{k})$$

Final Answer: $\vec{r} = (5 + 3\lambda)\hat{\imath} + (2 + 2\lambda)\hat{\jmath} + (-4 - 8\lambda)\hat{k}$

Q2. Find the Cartesian equation of the line passing through the points $(-1, 0, 2)$ and $(3, 4, 6)$.

Answer-

Direction ratios:

$$(3 - (-1),\ 4 - 0,\ 6 - 2) = (4,4,4)$$

Using two-point form:

$$\frac{x - (-1)}{4} = \frac{y - 0}{4} = \frac{z - 2}{4}$$
$$\frac{x + 1}{4} = \frac{y}{4} = \frac{z - 2}{4}$$

Divide throughout by 4:

$$\frac{x + 1}{1} = \frac{y}{1} = \frac{z - 2}{1}$$

Final Answer: $x + 1 = y = z - 2$

Q3. Find the angle between the lines whose direction ratios are proportional to $(1, 1, 2)$ and $(\sqrt{3} - 1,\ -\sqrt{3} - 1,\ 4)$.

Answer-

Let:

$$\vec{m}_1 = \hat{\imath} + \hat{\jmath} + 2\hat{k}$$
$$\vec{m}_2 = (\sqrt{3} - 1)\hat{\imath} + (-\sqrt{3} - 1)\hat{\jmath} + 4\hat{k}$$

Dot product:

$$\vec{m}_1 \cdot \vec{m}_2 = (1)(\sqrt{3}-1) + (1)(-\sqrt{3}-1) + (2)(4)$$
$$= \sqrt{3} - 1 - \sqrt{3} - 1 + 8 = 6$$

Magnitudes:

$$|\vec{m}_1| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{6}$$
$$|\vec{m}_2| = \sqrt{(\sqrt{3}-1)^2 + (-\sqrt{3}-1)^2 + 4^2}$$
$$= \sqrt{(4-2\sqrt{3}) + (4+2\sqrt{3}) + 16} = \sqrt{24} = 2\sqrt{6}$$
$$\cos\theta = \frac{6}{\sqrt{6}\cdot 2\sqrt{6}} = \frac{6}{12} = \frac{1}{2}$$
$$\theta = \cos^{-1}\left(\frac{1}{2}\right) = 60^\circ$$

Final Answer: $\theta = 60^\circ$ or $\frac{\pi}{3}$

Q4. Determine if the lines $\vec{r} = (\hat{\imath} - \hat{\jmath}) + \lambda(2\hat{\imath} + \hat{k})$ and $\vec{r} = (2\hat{\imath} - \hat{\jmath}) + \mu(\hat{\imath} + \hat{\jmath} - \hat{k})$ are perpendicular.

Answer-

Direction vectors:

$$\vec{m}_1 = 2\hat{\imath} + 0\hat{\jmath} + \hat{k}$$
$$\vec{m}_2 = \hat{\imath} + \hat{\jmath} - \hat{k}$$

Dot product:

$$\vec{m}_1 \cdot \vec{m}_2 = (2)(1) + (0)(1) + (1)(-1)$$
$$= 2 + 0 - 1 = 1$$

Since:

$$\vec{m}_1 \cdot \vec{m}_2 \neq 0$$

Final Answer: The lines are **NOT perpendicular**.

Q5. Find the shortest distance between the lines $\vec{r} = (\hat{\imath} + 2\hat{\jmath} + \hat{k}) + \lambda(\hat{\imath} - \hat{\jmath} + \hat{k})$ and $\vec{r} = (2\hat{\imath} - \hat{\jmath} - \hat{k}) + \mu(2\hat{\imath} + \hat{\jmath} + 2\hat{k})$.

Answer-

$$\vec{m}_1 = \hat{\imath} - \hat{\jmath} + \hat{k}, \vec{m}_2 = 2\hat{\imath} + \hat{\jmath} + 2\hat{k}$$

Difference:

$$\vec{a}_2 - \vec{a}_1 = \hat{\imath} - 3\hat{\jmath} - 2\hat{k}$$

Cross product:

$$\vec{m}_1 \times \vec{m}_2 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -1 & 1 \\ 2 & 1 & 2 \end{vmatrix} = -3\hat{\imath} + 0\hat{\jmath} + 3\hat{k}$$

Magnitude:

$$|\, \vec{m}_1 \times \vec{m}_2 \,| = \sqrt{9 + 0 + 9} = 3\sqrt{2}$$

Dot product:

$$(\vec{a}_2 - \vec{a}_1) \cdot (\vec{m}_1 \times \vec{m}_2) = (-3) + 0 + (-6) = -9$$

Distance:

$$d = \left|\frac{-9}{3\sqrt{2}}\right| = \frac{3}{\sqrt{2}}$$

Rationalizing:

$$d = \frac{3\sqrt{2}}{2}$$

Final Answer: $d = \frac{3\sqrt{2}}{2}$ units

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Linear Programming

Introduction

Linear programming is a mathematical technique used to allocate limited resources optimally to achieve a specific goal, such as maximizing profit or minimizing cost. It involves translating real-world decision-making problems into linear inequalities and solving them graphically to find the most efficient outcome.

Definitions of Various Terms Involved in Linear Programming

- **Objective Function:** A linear function of two or more variables which has to be maximized or minimized.

  Formula: $Z = ax + by$

- **Constraints:** The linear inequalities or equations restricting the variables representing the limited availability of resources.

- **Non-negative Restrictions:** The conditions indicating that the decision variables cannot be negative under any circumstances.

  Formula: $x \geq 0, \ y \geq 0$

- **Optimization Problem:** A problem that seeks to maximize or minimize a specific linear function subject to certain constraints.

- **Feasible Region:** The common region determined by all the constraints, including non-negative constraints, of a Linear Programming Problem.

- **Feasible Solution:** Any point located within or exactly on the boundary of the feasible region that satisfies all constraints simultaneously.

- **Optimal Solution:** Any valid feasible solution that provides the maximum or minimum value of the objective function.

Formulation of a Linear Programming Problem

Formulation is the mathematical process of translating a real-world word problem into algebraic equations and inequalities.

- **Step 1:** Identify the unknown decision variables to be determined and denote them precisely by x and y.
- **Step 2:** Identify the **Objective Function** and express it algebraically as a linear combination of x and y.
- **Step 3:** Identify all the **Constraints** based on the given conditions and express them as linear inequalities in terms of x and y.
- **Step 4:** Explicitly state the **non-negative restrictions** since physical quantities cannot be negative ($x \geq 0, y \geq 0$).

Solution of a Linear Programming Problem Graphically

Corner Point Method: A reliable graphical method to solve an LPP by evaluating the objective function only at the vertices of the feasible region.

- **Step 1:** Graph all the given linear inequalities on an XY-coordinate plane to find the common bounded **Feasible Region**.
- **Step 2:** Determine the exact coordinates of all the **Corner Points** (vertices) of this bounded feasible region.
- **Step 3:** Evaluate the objective function $Z = ax + by$ at each calculated corner point.
- **Step 4:** The maximum or minimum value of Z occurs precisely at one or more of these corner points. Choose the value based on the optimization goal.

TOP 5 QUESTIONS

Q1. Maximize $Z = 3x + 4y$ subject to the constraints $x + y \leq 4$, $x \geq 0$, and $y \geq 0$.

Answer-

Maximize

$$Z = 3x + 4y$$

subject to

$$x + y \leq 4, \quad x \geq 0, \quad y \geq 0$$

Step 1: Boundary Line

$$x + y = 4$$

Intercepts:

$$(4, 0), \quad (0, 4)$$

The line is drawn joining these two points.

Step 2: Feasible Region

Since

$$x + y \leq 4$$

and $x, y \geq 0$, the feasible region lies in the first quadrant below the line.

Corner points are:

$$(0, 0), \quad (4, 0), \quad (0, 4)$$

Step 3: Evaluate Objective Function

$$Z = 3x + 4y$$

$$Z(0, 0) = 0$$

$$Z(4, 0) = 12$$

$$Z(0, 4) = 16$$

Final Answer

$$\boxed{Z_{\max} = 16 \text{ at } (0, 4)}$$

Q2. Minimize graphically $Z = -3x + 4y$, subject to

$x + 2y \leq 8,$ $3x + 2y \leq 12,$ $x \geq 0, y \geq 0$

Answer-

Minimize

$$Z = -3x + 4y$$

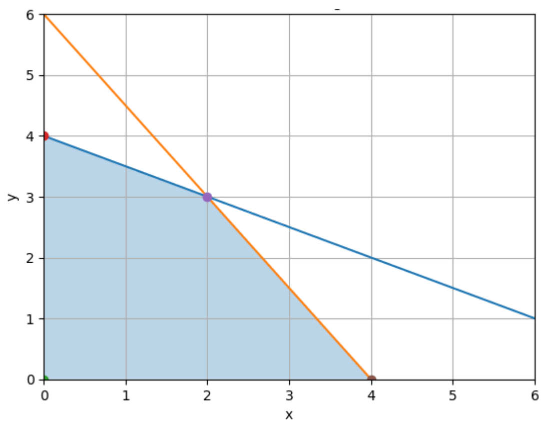
subject to

$$x + 2y \leq 8, \quad 3x + 2y \leq 12, \quad x \geq 0, \quad y \geq 0$$

Step 1: Boundary Lines

$$x + 2y = 8 \Rightarrow (0, 4), (8, 0)$$

$$3x + 2y = 12 \Rightarrow (0, 6), (4, 0)$$



Step 2: Feasible Region

Region lies in first quadrant satisfying both inequalities.
Intersection point of lines:

$$x + 2y = 8$$

$$3x + 2y = 12$$

Subtract:

$$2x = 4 \Rightarrow x = 2$$

$$y = 3$$

Corner points:

$$(0, 0), \quad (0, 4), \quad (2, 3), \quad (4, 0)$$

Step 3: Evaluate Objective Function

$$Z = -3x + 4y$$

$$Z(0,0) = 0$$

$$Z(0,4) = 16$$

$$Z(2,3) = -6 + 12 = 6$$

$$Z(4,0) = -12$$

Final Answer

$$\boxed{Z_{\text{min}} = -12 \text{ at } (4,0)}$$

Q3. Maximize $Z = 5x + 3y$ subject to

$$3x + 5y \leq 15, \quad 5x + 2y \leq 10, \quad x \geq 0, y \geq 0.$$

Answer-

Maximize

$$Z = 5x + 3y$$

subject to

$$3x + 5y \leq 15, \quad 5x + 2y \leq 10, \quad x \geq 0, \quad y \geq 0$$

Step 1: Boundary Lines

$$3x + 5y = 15 \Rightarrow (0,3),\ (5,0)$$

$$5x + 2y = 10 \Rightarrow (0,5),\ (2,0)$$

Step 2: Intersection Point

$$3x + 5y = 15$$

$$5x + 2y = 10$$

Multiply:

$$6x + 10y = 30$$

$$25x + 10y = 50$$

Subtract:

$$19x = 20 \Rightarrow x = \frac{20}{19}$$

Substitute:

$$5\left(\frac{20}{19}\right) + 2y = 10$$

$$\frac{100}{19} + 2y = 10$$

$$2y = \frac{90}{19} \Rightarrow y = \frac{45}{19}$$

Intersection point:

$$\left(\frac{20}{19}, \frac{45}{19}\right)$$

Step 3: Corner Points

$$(0,0), \quad (0,3), \quad (2,0), \quad \left(\frac{20}{19}, \frac{45}{19}\right)$$

Step 4: Evaluate Objective Function

$$Z(0,0) = 0$$

$$Z(0,3) = 9$$

$$Z(2,0) = 10$$

$$Z\left(\frac{20}{19}, \frac{45}{19}\right) = \frac{100+135}{19} = \frac{235}{19} \approx 12.37$$

Final Answer

$$\boxed{Z_{\max} = \frac{235}{19} \text{ at } \left(\frac{20}{19}, \frac{45}{19}\right)}$$

Q4. Formulate the mathematical constraints for the following: A machine takes 2 hours to make product A and 3 hours to make product B. The machine can work for a maximum of 12 hours.

Answer-

A machine takes 2 hours to make product A and 3 hours to make product B. Total available time = 12 hours.

Let

$$x = \text{units of product A}, \quad y = \text{units of product B}$$

Time constraint:

$$2x + 3y \leq 12$$

Non-negativity:

$$x \geq 0, \quad y \geq 0$$

Final Answer:

$$\boxed{2x + 3y \leq 12, \quad x \geq 0,\ y \geq 0}$$

Q5. Evaluate the objective function $Z = 2x + 5y$ at the corner points $(0, 5), (4, 3), and\ (0, 0)$ to find its maximum value.

Answer-

Evaluate

$$Z = 2x + 5y$$

at given corner points.

Given points:

$$(0, 5), \quad (4, 3), \quad (0, 0)$$

Compute values:

$$Z(0, 5) = 2(0) + 5(5) = 25$$

$$Z(4, 3) = 2(4) + 5(3) = 8 + 15 = 23$$

$$Z(0, 0) = 0$$

Final Answer:

$$\boxed{Z_{\max} = 25 \text{ at } (0, 5)}$$

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Mathematical Reasoning

Introduction

Mathematical reasoning involves formalizing the process of logical deduction. In this chapter, we will learn about mathematically acceptable statements, their negations, logical connectives like "and" and "or", quantifiers, and various methods to rigorously validate or formally prove mathematical statements.

Mathematical Statements

- A **Statement** (or proposition) is an assertive declarative sentence that is definitively either true or false, but never both simultaneously.
- Sentences that are imperative (commands), interrogative (questions), or exclamatory are not considered mathematical statements.
- Sentences involving variable time (like "today", "tomorrow") or variable places are also not statements unless their exact context is specified.

Negation of a Statement

- The **Negation** of a statement is the absolute denial of the original statement.
- If a statement is denoted by p, its negation is denoted by $\sim p$ (read as '$not\ p$').
- If p is exactly true, then $\sim p$ is strictly false, and vice versa.

Compound Statements

- A **Compound Statement** is formed by combining two or more simple statements using specific logical words or phrases.
- Each individual simple statement that makes up a compound statement is called a **Component Statement**.

Basic Logical Connectives

- **Conjunction ("And"):** A compound statement formed by connecting two simple statements with "And". It is true only if both component statements are exactly true.
- **Disjunction ("Or"):** A compound statement formed by connecting two simple statements with "Or". It is false only if both component statements are exactly false.
- **Exclusive "Or":** A condition where only one of the two statements can be true, but absolutely not both.
- **Inclusive "Or":** A condition where at least one of the statements is true, and possibly both are true.

Quantifiers

- **Quantifiers** are specific phrases like "There exists" and "For every" that quantify the variables present in a mathematical statement.
- **Existential Quantifier:** The phrase "There exists" (denoted by $\exists$) means there is at least one element for which the given condition holds true.
- **Universal Quantifier:** The phrase "For all" or "For every" (denoted by $\forall$) means the given condition holds true for every single element in the defined set.

Implications

- **Conditional ("If-Then"):** The statement "If p, then q" is denoted by $p \Rightarrow q$. Here, p is the sufficient condition for q.
- **Contrapositive:** The contrapositive of "If p, then q" is completely defined as "If $\sim q$, then $\sim p$". Both statements possess the exact same truth value.
- **Converse:** The converse of "If p, then q" is formed by reversing the order: "If q, then p".
- **Biconditional ("If and only if"):** Denoted by $p \Leftrightarrow q$. It means both $p \Rightarrow q$ and $q \Rightarrow p$ are true simultaneously.

Validating Statements

- **Direct Method:** To analytically prove "If p, then q", assume p is perfectly true and logically deduce that q must also be true.

- **Contrapositive Method:** Assume q is completely false (~q) and logically deduce that p must therefore be false (~p).
- **Method of Contradiction:** Assume that the given statement p is completely false, and mathematically deduce a contradiction, proving p must essentially be true.
- **By Counter-Example:** To definitively prove a general statement is false, finding just one specific example where the mathematical condition fails is strictly sufficient.

TOP 5 QUESTIONS

Q1. Check whether the following sentence is a statement: "The square of a real number is always positive."

Answer- It is a declarative sentence that is strictly false (since $0^2 = 0$, which is not strictly positive).

Because it has a definitive truth value (False), it is a valid mathematical statement.

Q2. Write the negation of the statement: "All prime numbers are odd."

Answer- To negate a universal quantifier ("All"), we use an existential quantifier ("There exists").

The negation is: "There exists at least one prime number which is not odd."

Q3. Write the contrapositive of the conditional statement: "If a number is divisible by 9, then it is divisible by 3."

Answer- The contrapositive reverses the order of the component statements and strictly negates both of them.

The contrapositive is: "If a number is not divisible by 3, then it is not divisible by 9."

Q4. Identify the component statements and the connective in the compound statement: "$\sqrt{2}$ is a rational number or an irrational number."

Answer- Component 1 (p): $\sqrt{2}$ is a rational number.

Component 2 (q): $\sqrt{2}$ is an irrational number.

The logical connective joining these components is "Or".

Q5. Use the method of counter-example to prove the statement false: "For every real number $x, x^2 > x$."

Answer- Assume $x = 0.5$.

Calculate x^2: $(0.5)^2 = 0.25$.

Since 0.25 is strictly less than 0.5, the condition $x^2 > x$ fails.

This single counter-example definitively proves the original statement is false.